CHAPTER 9

The influence of rotation, stratification, and magnetic fields on turbulence

9.1 The importance of body forces in geophysics and astrophysics

By and large, engineers do not have to worry too much about the influence of body forces on turbulence. Perhaps a little buoyancy crops up from time to time, but that is about it. The primary concern of the engineer is the influence of complex boundaries and the way in which these generate and shape the turbulence. The physicist, on the other hand, generally has to contend with flows in which body forces are the dominant factor. Astrophysicists, for example, might be concerned with the formation and evolution of stars, or perhaps with violent activity on the surface of the sun (solar flares, sun spots, coronal mass ejections, etc.). In either case turbulence plays a crucial role, transferring heat from the interior of a star to its surface, and triggering solar flares and coronal mass ejections. Moreover, this is a special kind of turbulence, shaped and controlled by intense magnetic fields. Geophysicists, on the other hand, might be interested in the motion of the earth's liquid core, and in particular, the manner in which turbulence in the core stretches and twists the earth's magnetic field in a way which prevents it from being extinguished through the natural forces of decay. Here the dominant forces acting on the turbulence are the Coriolis and Lorentz forces, arising from the earth's rotation and the terrestrial magnetic field, respectively. Indeed, the non-linear inertial force, $\mathbf{u} \cdot \nabla \mathbf{u}$, which has been the obsession of Chapters 1–8, is almost completely unimportant in geodynamo theory! Large-scale atmospheric and oceanic flows are also heavily influenced by body forces, in this case buoyancy and, at the very large scales, the Coriolis force.

In view of the difficulty of making predictions about conventional turbulence, it might be thought that the task of incorporating gravitational, Coriolis, and Lorentz forces into some coherent statistical model is so overwhelming as to be quite impractical. In a sense this is true. The equations of turbulence incorporating these forces are extremely complex. Curiously though, there are aspects of these

Turbulence on the surface of the sun (Encyclopaedia Britannica).

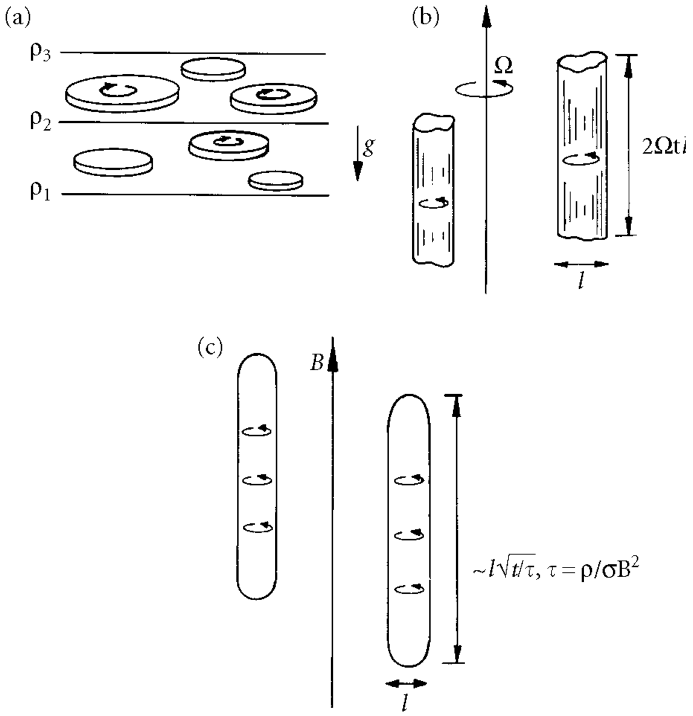

Figure 9.1 The type of large-scale eddies found in: (a) strongly stratified turbulence, (b) rapidly rotating fluid, and (c) a conducting fluid threaded by a magnetic field, **B**.

complex flows, that are easier to understand than conventional turbulence. The point is that buoyancy, Coriolis, and Lorentz forces all tend to organize and shape the turbulence, promoting vortices of a particular structure at the expense of other eddies. For example, it turns out that turbulence in a strongly stratified medium is dominated (at the large scales) by flat 'pancake' vortices (Figure 9.1(a)). A rapidly rotating fluid, on the other hand, tends to extrude vortices along the rotation axis forming columnar eddies (Plate 13; Figure 9.1(b)). Finally, a magnetic field causes vortices to diffuse along the magnetic field lines giving rise, once again, to columnar or sheet-like structures (Figure 9.1(c)). So, while the governing equations for these flows are messy and complex, the flows themselves tend to look more organized than conventional turbulence. The key to understanding turbulence in the presence of a body force is to isolate the mechanism by which that force organizes and shapes the motion. If this can be done, a great deal of useful information can be extracted from the analysis.

We shall look at the influence of rotation, stratification, and magnetic fields in turn, taking rotation and stratification together as they share many common characteristics. The discussion is brief, but the interested reader will find a more comprehensive discussion in the following texts and papers:

Rotational effects: Greenspan (1968), Cambon et al. (1997), and Iida and Nagano (1999).

Stratification: Panchev (1971), Monin and Yaglom (1971), Turner (1973), and Riley and Lelong (2000).

MHD turbulence: Moffatt (1978), Biskamp (1993), and Davidson (2001).

9.2 The influence of rapid rotation and stable stratification

We shall discuss the structure of turbulence in a rapidly rotating system in Sections 9.2.4 and 9.2.5, and turbulence in a stratified fluid in Section 9.2.6. First, however, we summarize some of the properties of the Coriolis force. In particular, we shall see that it tends to promote a form of internal wave motion, called an *inertial wave*. It is these waves which so dramatically shape the turbulent eddies in a rapidly rotating fluid. It turns out that inertial waves have a structure closely related to that of internal gravity waves, and it is this similarity in wave structure that underpins the close analogy between rotating and stratified turbulence. So let us start with the Coriolis force.

9.2.1 *The Coriolis force*

A frame of reference, which rotates at a constant rate Ω relative to an inertial frame is not inertial. The accelerations of a particle measured in the two frames are related by[1]

$$(d\mathbf{u}/dt)_{\text{inertial}} = (d\hat{\mathbf{u}}/dt)_{\text{rot}} + 2\boldsymbol{\Omega} \times \hat{\mathbf{u}} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \hat{\mathbf{x}}) \tag{9.1}$$

where $\hat{}$ indicates quantities measured in the rotating frame. The terms $2\boldsymbol{\Omega} \times \hat{\mathbf{u}}$ and $\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \hat{\mathbf{x}})$ are referred to as the Coriolis and centripetal accelerations, respectively. Equation (9.1) arises from applying the operator $(d/dt)_{\text{inertial}} = (d/dt)_{\text{rot}} + \boldsymbol{\Omega} \times$ to the radius vector $\mathbf{x}$, which gives $\mathbf{u} = \hat{\mathbf{u}} + \boldsymbol{\Omega} \times \hat{\mathbf{x}}$. Differentiating once more gives (9.1). If we multiply both sides of (9.1) by the mass of the particle, m, we have

$$m(d\mathbf{u}/dt)_{\text{inertial}} = m(d\hat{\mathbf{u}}/dt)_{\text{rot}} + m[2\boldsymbol{\Omega} \times \hat{\mathbf{u}}] + m[\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \hat{\mathbf{x}})].$$

Since the left-hand side of this equation is equal to $\mathbf{F}$, the sum of the forces acting on the particle, we can rewrite this as

$$m(d\hat{\mathbf{u}}/dt)_{\text{rot}} = \mathbf{F} - m[2\boldsymbol{\Omega} \times \hat{\mathbf{u}}] - m[\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \hat{\mathbf{x}})]$$

and we see that Newton's second law does not apply in the rotating frame. But we can 'fix' things if we add to the real forces $\mathbf{F}$, the fictitious forces $\mathbf{F}_{\text{cor}} = -m[2\boldsymbol{\Omega} \times \hat{\mathbf{u}}]$ and $\mathbf{F}_{\text{cen}} = -m[\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \hat{\mathbf{x}})]$. These are known as the Coriolis and centrifugal forces, respectively.

[1] For simplicity we shall take the two frames of reference to have a common origin so that $\mathbf{x} = \hat{\mathbf{x}}$. For a detailed discussion of rotating frames of reference, see, for example, Symon (1960).

Note that the centrifugal force is irrotational and may be written as $\nabla[\frac{1}{2}m(\boldsymbol{\Omega}\times\hat{\mathbf{x}})^2]$, that is,

$$\nabla[\tfrac{1}{2}(\boldsymbol{\Omega}\times\mathbf{x})^2] = (\boldsymbol{\Omega}\times\mathbf{x})\cdot\nabla(\boldsymbol{\Omega}\times\mathbf{x}) + (\boldsymbol{\Omega}\times\mathbf{x})\times[\nabla\times(\boldsymbol{\Omega}\times\mathbf{x})]$$
$$= \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{x}) + 2(\boldsymbol{\Omega}\times\mathbf{x})\times\boldsymbol{\Omega} = -\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{x}).$$

This is important in the context of fluid mechanics since the centrifugal force may be simply absorbed into the pressure term, $-\nabla p$, to form a modified pressure gradient. In the absence of a free surface, such forces produce no motion. So, introducing the modified pressure, $\hat{p} = p - \frac{1}{2}\rho(\boldsymbol{\Omega}\times\hat{\mathbf{x}})^2$, the Navier–Stokes equation in a rotating frame of reference becomes

$$\partial\hat{\mathbf{u}}/\partial t + \hat{\mathbf{u}}\cdot\nabla\hat{\mathbf{u}} = -\nabla(\hat{p}/\rho) + 2\hat{\mathbf{u}}\times\boldsymbol{\Omega} + \nu\nabla^2\hat{\mathbf{u}}. \tag{9.2}$$

From now on we shall omit the ^ on $\hat{\mathbf{u}}$, on the understanding that $\mathbf{u}$ is measured in the rotating frame, and the ^ on $\hat{p}$, on the understanding that p refers to the modified pressure. Note that, by necessity, the fictitious Coriolis force $2\mathbf{u}\times\boldsymbol{\Omega}$ cannot create or destroy energy, as evidenced by the fact that $(2\mathbf{u}\times\Omega)\cdot\mathbf{u} = 0$. Also, the relative strength of the non-linear inertial force, $\mathbf{u}\cdot\nabla\mathbf{u}$, and the Coriolis force, $2\mathbf{u}\times\boldsymbol{\Omega}$, is given by the so-called *Rossby number*, $\text{Ro} = u/l\Omega$, where l is a typical scale of the motion.

In the next few pages we shall omit the viscous term in (9.2) since the effect of the Coriolis force does not depend on viscosity. Also, to focus thoughts, we shall take $\boldsymbol{\Omega}$ to point in the z-direction and assume that Ro is small, so that the fluid is primarily in a state of rigid body rotation. Equation (9.2) becomes

$$\frac{D\mathbf{u}}{Dt} = 2\mathbf{u}\times\boldsymbol{\Omega} - \nabla(p/\rho). \tag{9.3}$$

Note that the Coriolis force tends to deflect a fluid particle in a direction normal to its instantaneous velocity, as illustrated in Figure 9.2. Thus, a fluid particle travelling radially outward experiences a force which tends to induce rotation in a sense opposite to that of $\boldsymbol{\Omega}$, so that its angular velocity measured in an inertial frame is reduced. Conversely, a particle moving radially inward will start to rotate (in the non-inertial frame) in the same sense as $\boldsymbol{\Omega}$. We might anticipate that, when viewed in an inertial frame, this curious behaviour is a direct consequence of the law of conservation of angular momentum and this is indeed more or less true. (Note, however, that individual fluid particles can exchange angular momentum via the pressure force and so this interpretation is a little simplistic. See Example 9.1.)

We shall see shortly that the Coriolis force has an extraordinary effect on a rotating fluid. In particular, it endows the fluid with a kind of elasticity, which allows it to propagate internal waves, called

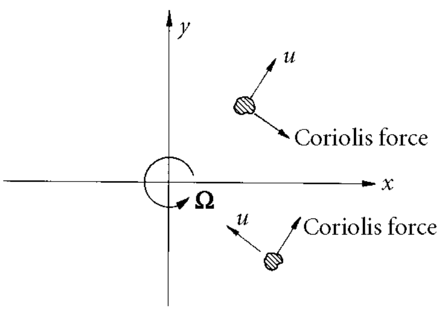

Figure 9.2 The influence of the Coriolis force on motion in the x–y plane.

inertial waves.[2] We can gain some insight into the origin of this phenomenon if we restrict ourselves to axisymmetric motion; that is, in our rotating frame of reference, we take $\mathbf{u}$ to be of the form $\mathbf{u}(r, z) = (u_r, u_\theta, u_z)$ in (r, θ, z) coordinates. Now suppose that, in our rotating frame, we have poloidal motion in the r–z plane, as shown in Figure 9.3(a). Initially the fluid has no relative rotation, $u_\theta = 0$. Fluid at A is swept inward to A' while fluid at B is carried outward to B'. This radial movement gives rise to a Coriolis force, $-2u_r\Omega\hat{\mathbf{e}}_\theta$, which induces positive relative rotation at A', $u_\theta > 0$, and negative relative rotation at B' (Figure 9.3(b)). Note that the direction of this induced rotation is such as to conserve angular momentum in an inertial frame of reference. The induced swirl itself now gives rise to a Coriolis force, $2u_\theta\Omega\hat{\mathbf{e}}_r$. This force opposes the original motion, tending to move the fluid at A' radially outward and the fluid at B' inward (Figure 9.3(c)). The whole process now begins in reverse and since energy is conserved in an inviscid fluid, we might anticipate that oscillations are set up, in which fluid particles oscillate about their equilibrium radii. These oscillations, which are the hallmark of a rapidly rotating fluid, are a manifestation of inertial wave propagation.

We shall return to inertial waves in Section 9.2.3, where we shall analyse their properties in detail. In the meantime, we consider another, closely related, consequence of rigid body rotation: the tendency for the Coriolis force to produce two-dimensional motion.

Example 9.1 Angular momentum conservation in inertial and non-inertial frames

Let us temporarily return to the use of the ^ to indicate variables in the rotating frame. Show that (9.3) yields

$$(D(\hat{\mathbf{x}} \times \hat{\mathbf{u}})/Dt)_{\text{rot}} = 2\hat{\mathbf{x}} \times (\hat{\mathbf{u}} \times \boldsymbol{\Omega}) + \nabla \times [(\hat{p}/\rho)\hat{\mathbf{x}}]. \tag{9.4}$$

The second term on the right-hand side integrates to zero for a localized disturbance, but the first need not. Evidently, angular momentum, as measured in the rotating frame, is not conserved. The first term on the right may be transformed using the identity:

$$\{2\mathbf{x} \times [\mathbf{v} \times \mathbf{K}]\}_i = \{(\mathbf{x} \times \mathbf{v}) \times \mathbf{K}\}_i + \nabla \cdot [\mathbf{x} \times (\mathbf{x} \times \mathbf{K})_i \mathbf{v}] \tag{9.5}$$

where $\mathbf{v}$ is any solenoidal vector field and $\mathbf{K}$ is a constant vector. By equating $\hat{\mathbf{u}}$ to $\mathbf{v}$ and $\boldsymbol{\Omega}$ to $\mathbf{K}$, show that (9.4) can be rearranged to give

$$(D/Dt)_{\text{rot}}[\hat{\mathbf{x}} \times (\hat{\mathbf{u}} + (\boldsymbol{\Omega} \times \hat{\mathbf{x}}))] + \boldsymbol{\Omega} \times [\hat{\mathbf{x}} \times (\hat{\mathbf{u}} + (\boldsymbol{\Omega} \times \hat{\mathbf{x}}))] \\ = \nabla \times [(p/\rho)\hat{\mathbf{x}}]. \tag{9.6}$$

[2] A particularly simple and beautiful explanation of inertial waves, which is different from ours and relies on an analogy between rotation and density stratification, is given by Rayleigh (1916).

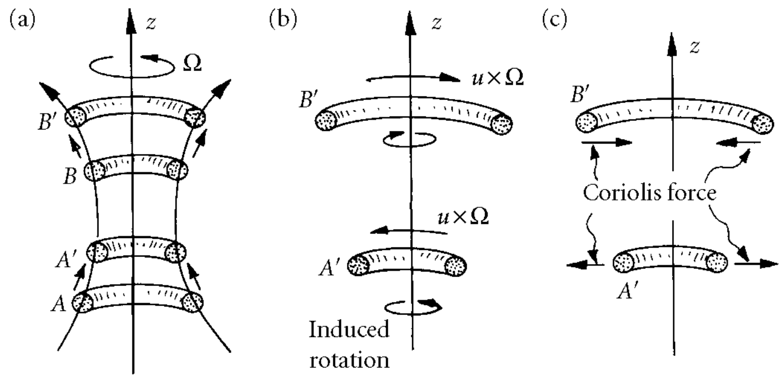

Figure 9.3 Sequence of events that leads to inertial waves through the action of the Coriolis force.

Since $\mathbf{u} = \hat{\mathbf{u}} + \boldsymbol{\Omega} \times \hat{\mathbf{x}}$, this simplifies to

$$[(D/Dt)_{\text{rot}} + \boldsymbol{\Omega}\times](\hat{\mathbf{x}} \times \mathbf{u}) = \nabla \times [(p/\rho)\hat{\mathbf{x}}]. \tag{9.7}$$

Finally show that this reduces to the inertial frame equation:

$$[(D/Dt)_{\text{inertial}}](\mathbf{x} \times \mathbf{u}) = \nabla \times [(p/\rho)\mathbf{x}] \tag{9.8}$$

which, unlike (9.4), does conserve angular momentum. (The term on the right integrates to zero for a localized disturbance.)

9.2.2 *The Taylor–Proudman theorem*

In a rotating frame of reference, our inviscid equation of motion is,

$$\frac{D\mathbf{u}}{Dt} = 2\mathbf{u} \times \boldsymbol{\Omega} - \nabla(p/\rho), \quad \boldsymbol{\Omega} = \Omega\hat{\mathbf{e}}_z. \tag{9.9}$$

We are particularly interested in cases where the departures from rigid-body rotation are slight, so the Rossby number, $u/l\Omega$, is small. In such cases the inertial term $\mathbf{u} \cdot \nabla\mathbf{u}$ may be neglected by comparison with the Coriolis force and we have

$$\frac{\partial\mathbf{u}}{\partial t} = 2\mathbf{u} \times \boldsymbol{\Omega} - \nabla(p/\rho). \tag{9.10}$$

We may eliminate pressure by taking the curl of (9.10). This provides us with a linearized vorticity equation:

$$\frac{\partial\boldsymbol{\omega}}{\partial t} = 2(\boldsymbol{\Omega} \cdot \nabla)\mathbf{u}. \tag{9.11}$$

If the motion is steady, or quasi-steady, we may neglect $\partial\boldsymbol{\omega}/\partial t$, which yields,

$$(\boldsymbol{\Omega} \cdot \nabla)\mathbf{u} = 0. \tag{9.12}$$

We have arrived at the *Taylor–Proudman theorem*. In cases where $u \ll \Omega l$ and $\partial\mathbf{u}/\partial t$ is small, the motion must be purely two dimensional,